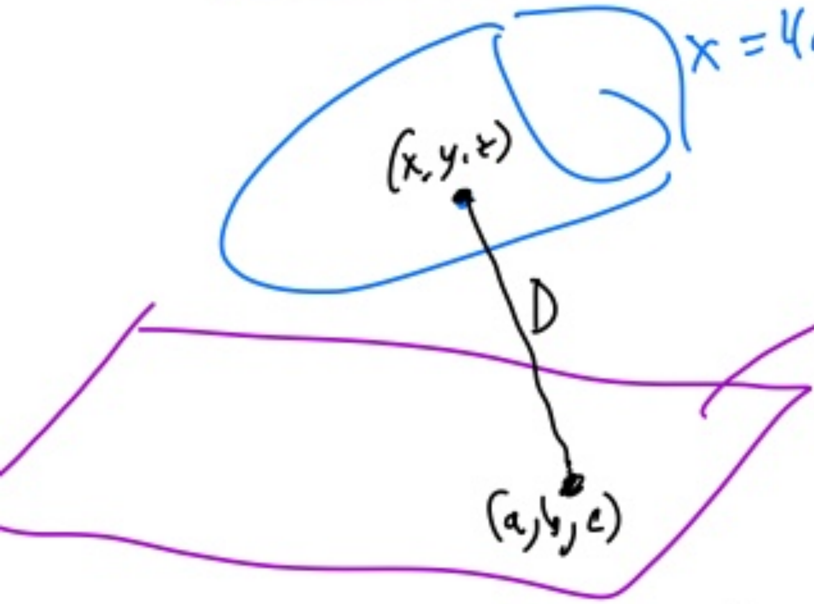
First question 2.51 (minimum) Distance between

$x = 4y^2 + z^2 + 2$

& $-x + y + 2z = 10$



$x = 4y^2 + z^2 + 2$

$g_1(x,y,z) = \boxed{x - 4y^2 - z^2 = 2}$

$-x + y + 2z = 10$

$g_2(a,b,c) = \boxed{-a + b + 2c = 10}$

$$F(x,y,z,a,b,c) = D^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

$$\nabla F = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$F_x = \lambda_1 (g_1)_x + \lambda_2 (g_2)_x$$

$$\boxed{\begin{aligned} 2(x-a) &= \lambda_1 \cdot 1 + \lambda_2 \cdot 0 \\ 2(y-b) &= \lambda_1 \cdot (-8y) + \lambda_2 \cdot 0 \\ 2(z-c) &= \lambda_1 \cdot (-2z) + \lambda_2 \cdot 0 \end{aligned}}$$

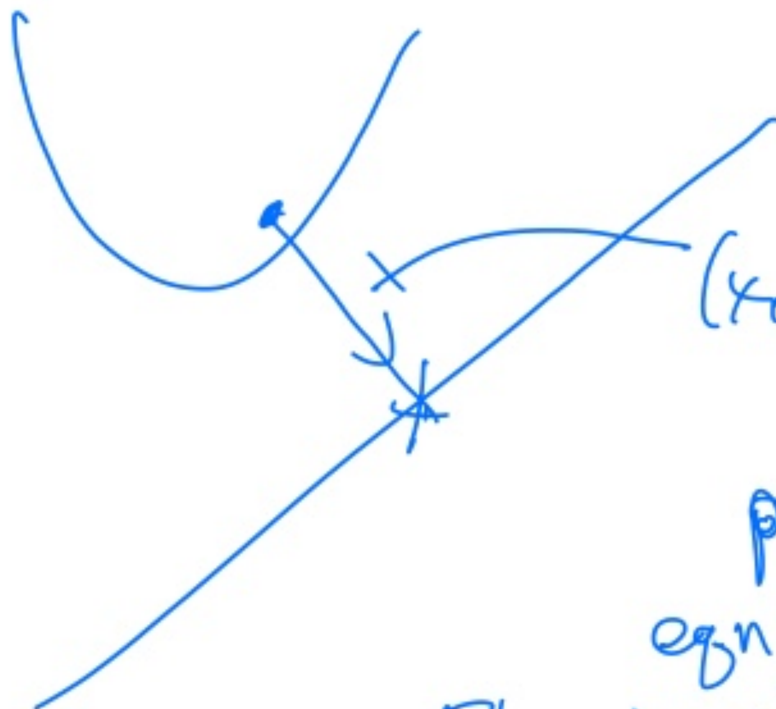
8 eqns
8 unknowns

$$F_a = \lambda_1 (g_1)_a + \lambda_2 (g_2)_a$$

$$\boxed{\begin{aligned} -2(x-a) &= \lambda_1 \cdot 0 + \lambda_2 (-1) \\ -2(y-b) &= \lambda_1 \cdot 0 + \lambda_2 (1) \\ -2(z-c) &= \lambda_1 \cdot 0 + \lambda_2 (2) \end{aligned}}$$

Easier way: $\nabla g_1 = \lambda \nabla g_2$ solves for (x, y, z)

$$\left.\begin{aligned} 1 &= \lambda(-1) \\ -8y &= \lambda(1) \\ -2z &= \lambda(2) \end{aligned}\right\} \text{ get } (x_0, y_0, z_0)$$



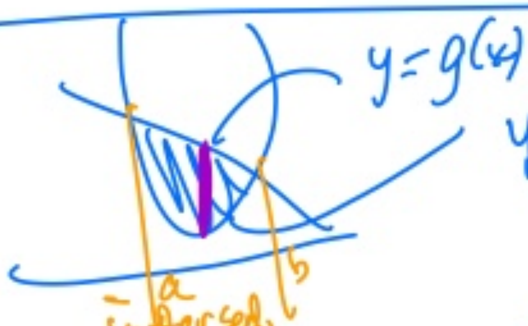
$(x_0, y_0, z_0) + t(-1, 1, 2)$

$\downarrow$

plug into to plane eqn, solve for t.

Plug in t to get (a, b, c) on plane.

3.2 c)



$$\int_{x=a}^{b} \int_{f(x)}^{g(x)} dy\, dx$$

Warm-up:

① Reverse the order of integration and compute:

$$\int_0^{\ln 2}\int_{e^y}^{2} y\,dx\,dy$$

Compute: $\displaystyle\int_0^{\ln 2}\left(\int_{e^y}^{2} y\,dx\right)dy$

$$= \int_0^{\ln 2}\left(yx\Big|_{x=e^y}^{x=2}\right)dy = \int_0^{\ln 2}(2y - ye^y)\,dy$$

$$\int (2y - ye^y)\,dy = y^2 - \int ye^y\,dy = y^2 - \left[ye^y - \underbrace{\int e^y\,dy}_{e^y}\right]$$

parts
$u = y \quad du = dy$
$dv = e^y dy \quad v = e^y$

$$= y^2 - ye^y + e^y$$

$$= y^2 - ye^y + e^y\Big|_0^{\ln 2}$$

$$= (\ln 2)^2 - (\ln 2)\underbrace{e^{(\ln 2)}}_{2} + \underbrace{e^{(\ln 2)}}_{2} - \left[0 - 0 + \underbrace{e^0}_{1}\right)$$

$$= (\ln 2)^2 - 2\ln 2 + 2 - 1$$

$$= \boxed{(\ln 2)^2 - 2\ln 2 + 1}$$

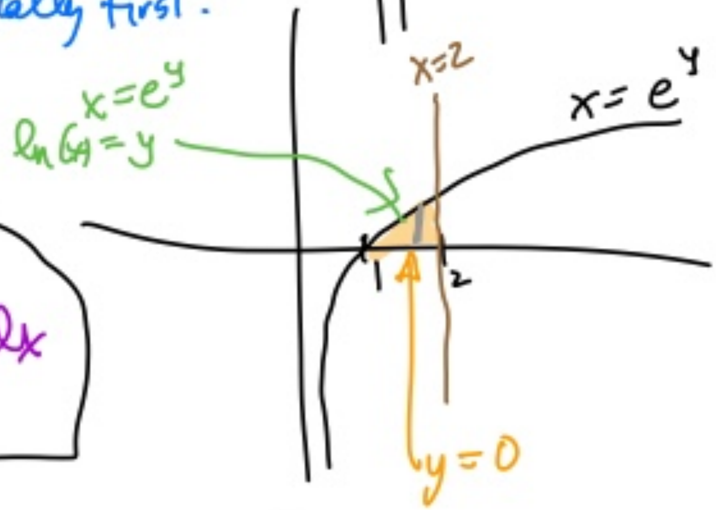
Reverse order: first graph

$$\int_0^{\ln 2}\int_{e^y}^{2} y\,dx\,dy$$

$x=e^y$ to $x=2$

x is integrated first (horizontal integral at fixed y)

Changing order: vertically first.



$$\boxed{\int_{x=1}^{2}\int_{y=0}^{\ln(x)} y\,dy\,dx}$$

$$\int_{x=1}^{2}\left(\tfrac{1}{2}y^2\Big|_0^{\ln(x)}\right)dx = \int_1^2 \tfrac{1}{2}(\ln(x))^2\,dx$$

$$= \tfrac{1}{2}\int_1^2 (\ln(x))^2\,dx$$

parts $u=(\ln(x))^2$ $du = 2\ln(x)\frac{1}{x}dx$

$dv = dx$ $v = x$

$$= \tfrac{1}{2}\left[x(\ln(x))^2 - 2\underbrace{\int \ln(x)\,dx}\right]$$

parts $u=\ln(x)$ $du=\frac{1}{x}dx$

$dv=dx$ $v=x$

$$= \tfrac{1}{2}x(\ln(x))^2 - \left[x\ln(x) - \underbrace{\int dx}_{x}\right]$$

$$= \frac{1}{2}x(\ln(x))^2 - x\ln(x) + x + C$$

$$= \frac{1}{2}x(\ln|x|)^2 - x\ln|x| + x \Big|_1^2$$

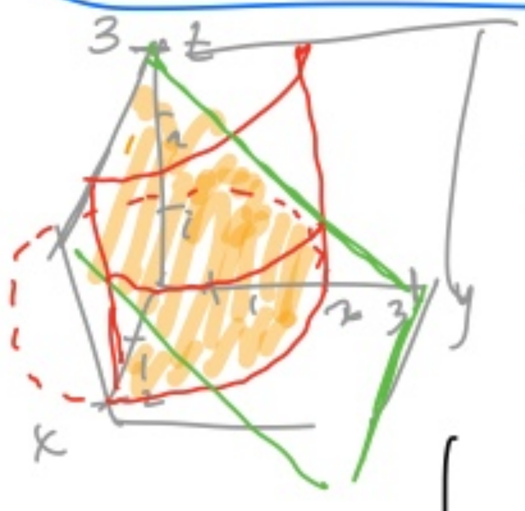
$$= \frac{1}{2}2(\ln(2))^2 - 2\ln(2) + 2$$

$$- \frac{1}{2}1(\underbrace{\ln(1)}_{0})^2 + 1\underbrace{\ln(1)}_{0} - 1$$

$$= \boxed{(\ln(2))^2 - 2\ln(2) + 1}$$

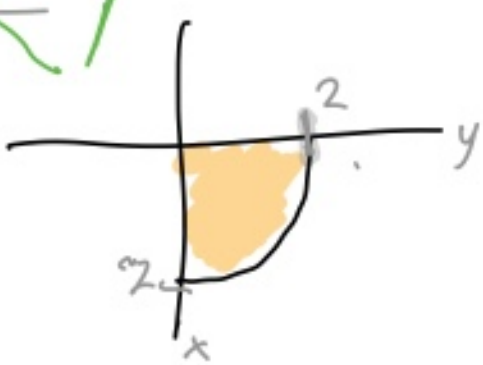
Example) Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$, and the plane $z+y=3$.

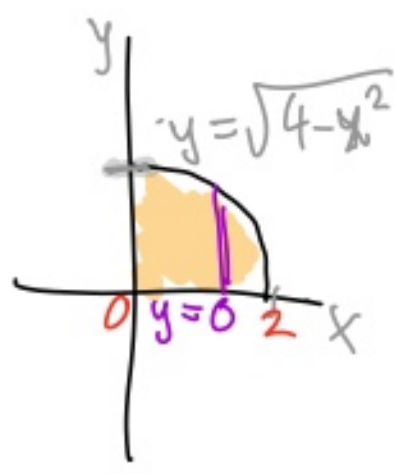
$z=3-y$



$$\text{Volume} = \iint_{R} (3-y)\,dx\,dy$$

volume under the surface $z=3-y$





$$\text{Volume} = \int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} (3-y)\,dy\,dx$$

$$= \int_{x=0}^{2} \left(3y - \tfrac{1}{2}y^2\right)\Big|_0^{\sqrt{4-x^2}}\,dx$$

$$= \int_0^2 \left[3\sqrt{4-x^2} - \tfrac{1}{2}(4-x^2)\right] dx$$

$$= 3\int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 2\,dx + \int_0^2 \tfrac{1}{2}x^2\,dx$$

trig sub
$x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

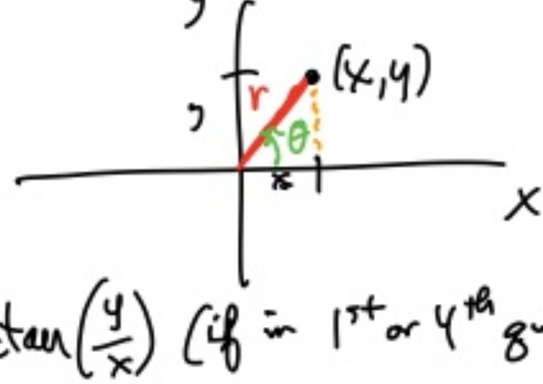
$$= 3\cdot\tfrac{1}{4}(\pi\cdot 2^2) - 2x\Big|_0^2 + \tfrac{1}{6}x^3\Big|_0^2$$

$$= 3\pi - 4 + \cancel{\tfrac{8}{6}}\ \tfrac{4}{3}$$

$$= \boxed{3\pi - \tfrac{8}{3}}.$$

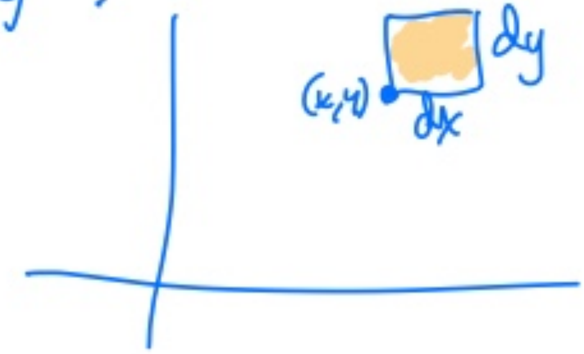
Polar Coordinates (r,θ) instead of (x,y)

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$



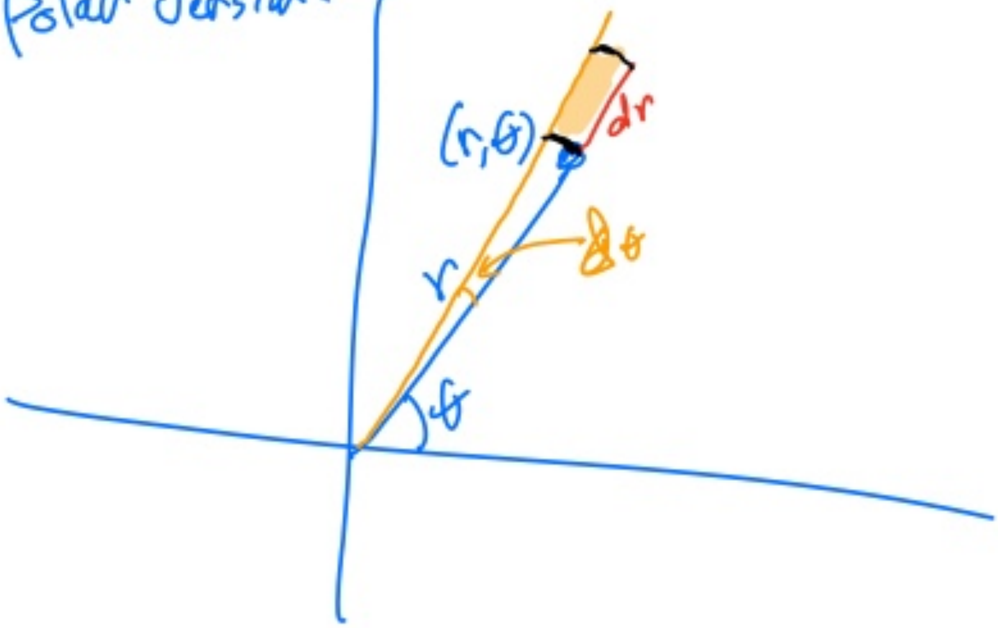
$r = \sqrt{x^2+y^2}$ $\quad\theta = \arctan\left(\frac{y}{x}\right)$ (if in 1st or 4th quadrant)

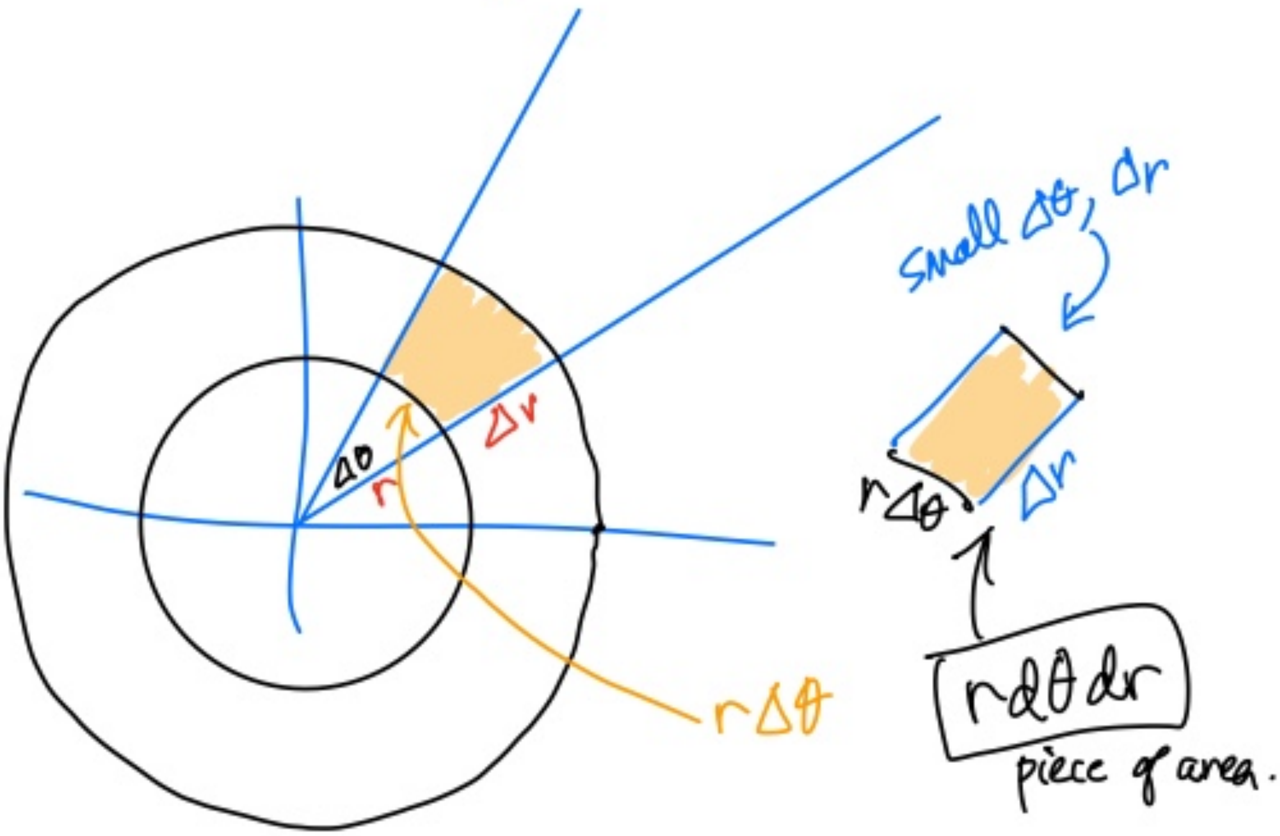
What about $dx\,dy$?



infinitessimal piece of area.

Polar version.





ie $$\iint_R f(x,y)\,\boxed{dx\,dy} = \iint_R \underbrace{\tilde{f}(r,\theta)}_{f(r\cos\theta,\, r\sin\theta)}\,\boxed{r\,d\theta\,dr}$$

Another way to compute the area element — uses differential forms

$$f \Rightarrow df = f_x\,dx + f_y\,dy + \cdots$$

$$d(f\,dx) = (df)\wedge dx$$
$$= (f_x\,dx + f_y\,dy)\wedge dx$$

wedge product

rules: ① $dx \wedge dx = 0$

$dy \wedge dy = 0$

$\vdots$

② $dx \wedge dy = -dy \wedge dx$

anticommutative

$$d(f\,dx) = (f_x\,dx + f_y\,dy)\wedge dx$$
$$= f_x\,dx\wedge dx + f_y\,dy\wedge dx$$
$$\overset{②}{=} \boxed{-f_y\,dx\wedge dy}$$

$dx \wedge dy$ oriented area element.

Coordinate change: $x = r\cos\theta$
$y = r\sin\theta$

$$dx = d(r)\cos\theta + r\,d(\cos\theta)$$
$$= \cos\theta\,dr + r(-\sin\theta\,d\theta)$$
$$= \cos\theta\,dr - r\sin\theta\,d\theta$$

$$dy = d(r\sin\theta) = dr(\sin\theta) + r\,\underbrace{d(\sin\theta)}_{\cos\theta\,d\theta}$$
$$= \sin\theta\,dr + r\cos\theta\,d\theta$$

$$dx \wedge dy = (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge(\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= \underbrace{\cos\theta\sin\theta\,dr\wedge dr}_{0} - r\sin^2\theta\,\underbrace{d\theta\wedge dr}_{-dr\wedge d\theta} + r\cos^2\theta\,dr\wedge d\theta - r^2\sin\theta\cos\theta\,\underbrace{d\theta\wedge d\theta}_{0}$$

$$= (r\sin^2\theta + r\cos^2\theta)\,dr\wedge d\theta$$

$$= r(\sin^2\theta + \cos^2\theta)\,dr\wedge d\theta = r\,dr\wedge d\theta$$

$$\boxed{dx\wedge dy = r\,dr\wedge d\theta}$$